Renormalized coupling constants and related amplitude ratios for Ising systems

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Using modern estimates and exact values of critical points and exponents for Ising systems, existing series expansions for various lattices are analyzed to estimate the universal amplitude ratio $R_0 \equiv (C_4^+)^2/C^+C_6^+$, where $C^{\pm}$ are the amplitudes of divergence of the susceptibility χ, above and below T_c, while $C_k^{\pm}$ are, similarly, the amplitudes for $(\partial^{k-2}\chi/\partial h^{k-2})_{h=0}$; we obtain $R_0 = 0.1582 \pm 0.0002$ for $d=2$ dimensions and 0.1275 ± 0.0003 for $d=3$. Similarly, we find the universal ratio $C_4^+/C_4^- = -9.0 \pm 0.3$ and -121.5 ± 0.5 for $d=3$ and 2, respectively. On using existing estimates for C^+, for the second-moment correlation-length amplitude f_1^+, and for the universal ratios C^+/C^- and f_1^+/f_1^-, we estimate the renormalized coupling constants to be $g_+^* = G_1^+ \equiv -C_4^+/(C^+)^2(f_1^+a)^d/v_0 = 24.45 \pm 0.15$ for $d=3$ and 14.700 ± 0.017 for $d=2$. Beneath T_c we find $G_1^- = -501 \pm 60$, -1768 ± 80 and $-C_3^- B/(C^-)^2 = 6.4_7 \pm 0.2$, $33.0_6 \pm 0.1$, where B is the spontaneous-magnetization amplitude, and thence $g_-^* \simeq 85$ and 23 for $d=3$ and 2, respectively. [S1063-651X(96)04008-1]

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I. INTRODUCTION

Perturbative expansions of ϕ^4 field theory at fixed dimension $d<4$ have been studied to calculate critical exponents such as α,β, etc. [1,2]. An important parameter that enters the calculations is the renormalized coupling constant g. In the critical region, the bare coupling constant g_0 of the initial Hamiltonian becomes infinite on the scale fixed by the correlation length [3,4], whereas the renormalized coupling constant g will approach a finite nonzero limit g^* at criticality provided that hyperscaling holds [4–6]. Even though the renormalized coupling constant g^* plays an important role, it seems not to have been analyzed by series extrapolation techniques in light of recently improved knowledge of critical points and exponents for Ising systems [7]. Accordingly, we have undertaken to estimate $g^* \equiv g_+^*$ and also g_-^*, the renormalized coupling constant beneath T_c that has a quite distinct value, as well as various related universal amplitude ratios for Ising lattices; we use existing long series expansions for various lattices together with modern estimates of exponents and critical temperatures.

To be more explicit, we employ the reduced temperature variable

$$t \equiv (T-T_c)/T_c \tag{1.1}$$

and the reduced field

$$h \equiv \mu_0 H/k_B T, \tag{1.2}$$

where μ_0 is the magnetic moment per Ising spin $s_i = \pm 1$. For $h=0$, the reduced susceptibility diverges as

$$\chi = (\partial m/\partial h)_T/\mu_0 \approx C^{\pm}|t|^{-\gamma}, \quad t \to 0\pm, \tag{1.3}$$

where $m(T,h)$ is the mean magnetization per spin. Likewise, the higher field derivatives behave as

$$\frac{\partial \chi}{\partial h} \approx C_3^{\pm}|t|^{-\gamma-\Delta}, \quad \frac{\partial^2 \chi}{\partial h^2} \approx C_4^{\pm}|t|^{-\gamma-2\Delta}, \tag{1.4}$$

etc., when $t \to 0\pm$, where, via the scaling relations, we have $\Delta = \beta + \gamma = 1 + \frac{1}{2}(\gamma - \alpha)$. The second-moment correlation length $\xi_1(T)$ [8] diverges as

$$\xi_1 \approx f_1^{\pm} a |t|^{-\nu}, \quad t \to 0\pm, \tag{1.5}$$

where a is the lattice spacing, while the spontaneous magnetization varies as

$$m_0/\mu_0 \equiv \langle s \rangle_0 \approx B|t|^{\beta}. \tag{1.6}$$

Now, according to scaling and universality concepts, the dimensionless amplitude ratios f_1^+/f_1^-, C^+/C^-, and C_4^+/C_4^- are universal, depending only on d (for Ising systems). Watson [9] seems to have been the first author to stress this and provide examples. Two other universal dimensionless combinations, involving amplitudes defined only above or only below T_c, are [9,10] the sixth-order and third-order ratios

$$R_0 = (C_4^+)^2/C^+C_6^+, \quad R_3 = -C_3 B/(C^-)^2. \tag{1.7}$$

Finally, the renormalized coupling constant g_+^* can be defined by [3,4,6]

$$g_+^* = -\lim_{t \to 0+} \left. \frac{(\partial^2\chi/\partial h^2)}{\chi^2 \xi_1^d / v_0} \right|_{h=0}, \tag{1.8}$$

where v_0 is the volume per lattice site. In terms of the critical amplitudes, the renormalized coupling constant can be written

$$g_+^* = G_1^+ \equiv -C_4^+/(C^+)^2 (f_1^+)^d (a^d/v_0), \tag{1.9}$$

where the factor a^d/v_0 enters to account for the different lattice geometries. It takes the value 1 for the plane square (sq) lattice and $2/\sqrt{3}$ for the triangular (tri) lattice. In $d=3$, one has $a^d/v_0 = 1$, $\frac{3}{4}\sqrt{3}$, and $\sqrt{2}$ for the simple cubic (sc),

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TABLE I. Summary of the critical values and amplitudes for Ising lattices. For $d=3$, the critical exponents adopted are $\nu=0.6320$ and $\gamma=1.2395$. All other exponents used follow from the scaling and hyperscaling relations. The values and uncertainties of the critical points v_c [$\equiv\tanh(J/k_BT_c)$] are those of Liu and Fisher [7]; the amplitudes B, C^+, and f_1^+ are derived from their estimates by interpolation. The asterisk indicates that the exactly known *true* correlation length amplitude f^+ is quoted in place of f_1^+.

	sq	tri	sc	bcc	fcc
v_c	$\sqrt{2}-1$	$2-\sqrt{3}$	0.218 071 $\pm$0.000 012	0.156 082 $\pm$0.000 007	0.101 709 $\pm$0.000 005
B	1.222 410	1.203 270	1.68_5 $\pm 0.03_5$	1.61 $\pm 0.02_5$	1.591 $\pm$0.007
C^+	0.962 582	0.9242_2 $\pm$0.0002	1.0928 $\pm$0.0010	1.0216 $\pm$0.0008	1.0034 $\pm$0.0005
C_3^-	-0.0176_3 $\pm$0.0005	$-$0.016 52 $\pm$0.000 05	$-$0.186 $\pm$0.002	$-$0.171 $\pm$0.002	$-$0.168 $\pm$0.003
C_4^+	$-$4.3788 $\pm$0.0003	$-$4.0001 $\pm$0.0005	$-$3.630 $+$0.003 $-$0.017	$-$3.236 $\pm$0.002	$-$3.1704 $\pm$0.0012
C_4^-	0.0360 $\pm$0.0001	0.0330 $\pm$0.0002	0.40 $\pm$0.03	0.36_5 $\pm$0.03	0.357 $\pm$0.008
f_1^+	0.567 296*	$0.525\,526_4$*	0.4984 $+$0.0010 $-$0.0050	0.4608 $\pm$0.0002	0.4496 $\pm$0.0006

body-centered-cubic (bcc), and face-centered-cubic (fcc) lattices, respectively [7]. By using the amplitudes beneath T_c, one can define G_1^- in the same way and it should also be universal.

However, below T_c, the renormalized coupling constant g_-^* differs from G_1^- owing to the broken symmetry. In conception, the renormalized coupling constant is closely related to the fourth derivative of the ''Helmholtz free energy'' $A(T,m)$ with respect to the order parameter m since $A(T,m)$ resembles the underlying field-theoretic or Landau-Ginzburg-Wilson Hamiltonian most closely. Thus, by relating $(\partial^4 A/\partial m^4)$ to appropriate field derivatives and replacing $(\partial^2\chi/\partial h^2)$ in (1.8) by its counterpart, one can define g^* below criticality via

$$g_-^* = -\lim_{t\to 0-} \left.\frac{(\partial^2\chi/\partial h^2)-3(\partial\chi/\partial h)^2/\chi}{\chi^2\xi_1^d/v_0}\right|_{h=0}. \quad (1.10)$$

Note that, above T_c, the derivative $(\partial\chi/\partial h)$ vanishes identically when $h=0$, so this expression, in effect, reduces to (1.8). In terms of the critical amplitudes below T_c, this yields

$$g_-^* = -[C_4^- - 3(C_3^-)^2/C^-]/(C^-)^2(f_1^-)^d(a^d/v_0). \quad (1.11)$$

It turns out (via our results as summarized in Table III) that g_-^* is positive, whereas G_1^- is negative. Naturally, for reasons of stability, one expects the renormalized constant to be positive.

Of course, the critical exponents and temperatures are exactly known for $d=2$ Ising systems. We will use

$$\gamma=1\tfrac{3}{4}, \quad \Delta=1\tfrac{7}{8},$$

$$v_c\equiv\tanh(J/k_BT_c)=\sqrt{2}-1 \quad (\text{sq}), \quad (1.12)$$

$$v_c=2-\sqrt{3} \quad (\text{tri}).$$

For $d=3$ Ising systems the corresponding parameters are now rather well known as a result of series extrapolation and Monte Carlo and renormalization-group studies [7,11]. The values we have adopted are summarized in Table I. The leading correction-to-scaling term is important in estimating various critical parameters including the amplitudes. Recent work [11–14] indicates that the corresponding correction exponent is given by

$$\theta=0.54\pm 0.03. \quad (1.13)$$

To estimate the various critical amplitudes from series data, we have employed inhomogeneous differential approximants (DAs). A particular DA for a given series is denoted $[K/L;M]$, where K, L, and M are degrees of the polynomials that enter into the definition of the approximant: details of the approach used are given in Ref. [7]. It has also proved convenient to use direct Padé approximants $[L/M]$, which, however, do not allow effectively for explicit background terms.

In the following we first review, in Sec. II, the original estimates for R_0 obtained by Watson [9] and the predictions of the linear model [10]. Then we analyze currently available Ising model series data to estimate R_0 for $d=2$ and 3. In Sec. III the critical amplitudes $C_4^\pm$ are studied for sq, tri, sc, bcc, and fcc lattices: these yield estimates for the ratio C_4^+/C_4^- and confirm its universality. In Sec. IV the amplitude C_3^- is estimated for all these lattices; it is then used for studying the

TABLE II. Estimates for the sixth-order amplitude ratio R_0 for the bcc lattice obtained via direct Padé approximants evaluated at the critical point specified in Table I.

$[L/M]$	R_0	$[L/M]$	R_0	$[L/M]$	R_0
[4/7]	0.12756	[4/8]	0.12750	[4/9]	0.12744
[5/6]	0.12741	[5/7]	0.12720	[5/8]	0.12738
[6/5]	0.12748	[6/6]	0.12708	[6/7]	0.12733
[7/4]	0.12748	[7/5]	0.12745	[7/6]	0.12738
		[8/4]	0.12736	[8/5]	0.12716
				[9/4]	0.12721

third-order amplitude ratio R_3. Recent estimates are combined with the up-to-date values for C^+ and f_1^+ on various $d=3$ lattices in Sec. V to give estimates for the renormalized coupling constants $g_{\pm}^*$. The results are summarized briefly in Sec. VI.

II. SIXTH-ORDER AMPLITUDE RATIO

As mentioned, Watson [9] noted that certain combinations of critical amplitudes should be universal and for the exactly soluble *spherical model* he found

$$R_0=\begin{cases}\frac{6}{55}=0.109\,09\ldots & (d=3)\\ \frac{1}{10} & (d\geqslant 4),\end{cases} \tag{2.1}$$

where the spherical model value for $d\geqslant 4$ also follows from mean field theory. Then he used the individual amplitudes C^+, C_4^+, and C_6^+ for the Ising model on various lattices as estimated by Essam and Hunter [15]. This supported the universality hypothesis for Ising systems and led to

$$R_0\simeq\begin{cases}0.156 & (d=2)\\ 0.129 & (d=3).\end{cases} \tag{2.2}$$

It is also instructive to compute the universal ratio R_0 by using the so-called linear model of Schofield [16–18], which is a specific approximate, parametric representation [18] of the Ising equation of state in the scaling region. In the linear model t, h, and m are related via

$$t=r(1-b^2\theta^2),\quad h=r^{\Delta}l_0\theta(1-\theta^2),\quad m=r^{\beta}m_0\theta, \tag{2.3}$$

where $b>1$ and l_0 and m_0 are positive constants. The parameter r measures distance from criticality, while θ is a pseudopolar angle in the (t,m) or (t,h) planes. Although the linear model is not exact, it embodies the scaling laws and correct analytic behavior in all limits. Furthermore, it agrees precisely with the ϵ expansion to order ϵ^2 [19].

After some labor [10], one finds that the sixth-order amplitude ratio for the linear model can be written

$$R_0=\frac{3(1-b^2\gamma)^2}{5[2(1-b^2\gamma)(3-2b^2\Delta)+b^4\gamma(\gamma-1)]}. \tag{2.4}$$

For the mean field exponents $\gamma=\Delta-\beta=1$ and $\Delta=3/2$, R_0 is seen to be independent of the parameter b, and the mean field result $R_0=\frac{1}{10}$ is recovered. Otherwise, R_0 depends on b. To proceed, one may adopt the value

$$b^2=(\delta-3)/(\delta-1)(1-2\beta)\quad\text{where}\quad\delta=\Delta/\beta, \tag{2.5}$$

which was recommended by Schofield, Litster, and Ho [17]. At this value of b the predicted ratio C^+/C^- passes through a minimum for fixed exponents. This choice leads to

$$R_0=\frac{3\gamma(1-\gamma)}{5[(2-\alpha)^2+2\gamma(2\alpha-\gamma-1)]}. \tag{2.6}$$

Surprisingly, this also proves to be an extremum function of b; but it is a maximum rather than a minimum. Using the exact $d=2$ values and the estimates for $d=3$ in Table I, the special linear model thus predicts

$$R_0=0.140\,00\quad(d=2)$$
$$\simeq 0.1237\quad(d=3). \tag{2.7}$$

These values correlate quite well with (2.2). The $\sim$11% deviation for $d=2$ is not surprising since the linear model is known to fail for $d\lesssim 3$.

Since Essam and Hunter's work in 1968, longer Ising model series have become available [20,21]: in terms of the standard high-temperature variable $v\equiv\tanh(J/k_BT)$, these are known to order v^{14} for the plane triangular lattice and to order v^{17} for the square lattice. For $d=3$, the expansions have been derived to orders v^{17}, v^{13}, v^{10}, and v^{19} for the sc, bcc, fcc, and diamond (dia) lattices, respectively.

To estimate R_0 we form the series for the ratio function

$$\mathcal{R}(T)\equiv[(\partial^2\chi/\partial h^2)_0]^2/\chi(T)(\partial^4\chi/\partial h^4)_0, \tag{2.8}$$

which should exhibit the critical behavior

$$\mathcal{R}(T)=R_0(1+r_\theta t^\theta+r_1t+\cdots) \tag{2.9}$$

when $t\to 0+$. Since $\mathcal{R}(T)$ remains finite at T_c, it is reasonable to examine near-diagonal direct Padé approximants $[L/M]$ to the series. Data for the bcc lattice with $L+M=11$, 12, and 13 are displayed in Table II. Evidently, the values are quite well converged.

The other lattices exhibit equally good behavior except that occasional approximants contain ''tears,'' i.e., close-by zero-pole pairs in the physical region $v\lesssim v_c$; such defective approximants are discarded since, in general, they are of lower reliability. To summarize, the last three orders for the sc lattice generate approximants lying in the ranges 0.127 83 $\pm$5, 0.127 84$\pm$9, and 0.127 98$\pm$18 to order 17, where, in this section, the uncertainties quoted refer to the last decimal place of the central value. For the fcc lattice we find 0.127 97 $\pm$40, 0.127 53$\pm$22, and 0.127 53$\pm$4 to order 10 and for the diamond lattice with $v_c=0.353\,81$ [15], 0.127 61$\pm$83,

TABLE III. Universal amplitude ratios. The ratio f_1^+/f_1^- in $d=2$ is quoted from Ref. [24]. In three dimensions, the estimates for C^+/C^- and f_1^+/f_1^- are those of Ref. [7]. For the mean-field theory (MFT) results, see Sec. V.

Ratio	$d=2$	$d=3$	MFT
C^+/C^-	37.693 562	4.95 ± 0.15	2
$R_3=-C_3^-B/(C^-)^2$	$33.0_6\pm0.1$	$6.4_7\pm0.2$	3
C_4^+/C_4^-	-121.5 ± 0.5	-9.0 ± 0.3	-2
$R_0=(C_4^+)^2/C^+C_6^+$	0.1582 ± 0.0002	0.1275 ± 0.0003	$\frac{1}{10}$
f_1^+/f_1^-	3.22 ± 0.08	1.96 ± 0.01	$\sqrt{2}$
g_+^*	14.700 ± 0.017	24.45 ± 0.15	128
g_-^*	23	85	128
G_1^-	-1768 ± 80	-501 ± 60	-1024

$0.127\,81\pm47$, and $0.127\,87\pm37$, to order 17. By examining the full tables and the trends, one sees that universality is rather well confirmed. Explicitly, for the individual lattices, we estimate

$$R_0=\begin{cases}0.1275\pm1 & (\text{fcc})\\ 0.1274\pm2 & (\text{bcc})\\ 0.1279\pm3 & (\text{sc})\\ 0.1280\pm5 & (\text{dia}).\end{cases} \tag{2.10}$$

Overall, accepting universality, we adopt the value entered in Table III.

For the planar lattices, one finds ranges $0.158\,04\pm16$, $0.158\,49\pm46$, and $0.158\,06\pm10$ for the square lattice to order 17 and $0.158\,13\pm8$, $0.158\,16\pm6$, and $0.158\,18\pm2$ for the triangular lattice to order 14. Universality is again confirmed: our overall estimate is given in Table III.

It is appropriate to test the sensitivity of the estimates found to changes in the assumed critical points. In essence, this also checks the magnitude and significance of the correction terms in (2.9). To this end, Table II has been recomputed using the value $(J/k_BT_c)=0.157\,408$ [10], which actually lies well outside the range stated in Table I. The resulting changes are very small: specifically, each entry drops by one digit in the last decimal place. Since this is less by a factor 10 or more than the changes from approximant to approximant, the effect is quite negligible. For other lattices the influence of the precise value of the critical point is comparably small.

In principle, if the coefficient a_θ in (2.9) does not vanish, the leading singular behavior of the ratio function $\mathcal{R}(T)$ is $\sim(T-T_c)^\theta$ and that should be detectable by DA analysis. That, in turn, would allow for the effects of the nonanalytic correction and yield improved estimates of R_0 from the ''background'': see [7]. In practice, however, the amplitude a_θ seems to be too small to be detectable in this way. Indeed, unbiased approximants yield a wide range of critical point estimates and a corresponding range of θ estimates. Even when the preferred critical point is imposed, the distribution of θ estimates is broad and indefinite; the corresponding estimates for R_0 are quite consistent with the direct Padé data (as in Table II), but the spread is much larger. In short, as already indicated by the insensitivity to the assigned critical point, the leading singular term in the expansion (2.9) is either intrinsically small or is effectively cancelled by a combination of analytic and higher order contributions. No advantage accrues by attempting to allow for it.

Finally, we remark that our estimates differ from Watson's in (2.2) by only $+1.4\%$ and -1.2% for $d=2$ and 3, respectively, while the special linear model value for $d=3$ is about 3% low.

III. FOURTH-ORDER SUSCEPTIBILITY AMPLITUDE RATIO

Mean-field theory leads to the ratio

$$C_4^+/C_4^-=-2 \tag{3.1}$$

for the fourth-order susceptibilities above and below T_c. Owing to the presence of ''spin-wave'' modes in the spherical model for $d\leq4$, the susceptibility $\chi(T,h)$ diverges as $h\to0$ for $T<T_c$ and thus it is not useful to consider the ratio. However, the linear model leads to

$$\frac{C_4^+}{C_4^-}=\frac{12(1-\gamma b^2)(b^2-1)^{3+\beta-3\Delta}}{6[1+(2\Delta-3)b^2]+(3+\beta-3\Delta)[3\alpha+(1-2\beta)(2\gamma-\alpha)b^2]b^4}, \tag{3.2}$$

which, adopting Schofield's choice (2.5), gives

$$\frac{C_4^+}{C_4^-}=\frac{48(\gamma-1)^4(1-2\beta)^{-2}[2(\gamma-1)/(1-2\beta)(\delta-1)]^{\beta(1-3\delta)}}{8(1-\beta)(2\beta-3)+12(5-4\beta)\gamma-\gamma(\delta-1)[6(3+4\beta)+(\delta-1)(3\gamma-20\beta-1)]}. \tag{3.3}$$

This again represents an extremum, actually a minimum in magnitude as b varies in (3.2). Using the exact exponents and Table I thereby leads to the estimates

$$C_4^+/C_4^- = -\frac{4802\sqrt{7}}{11} \simeq -1154.99 \quad (d=2)$$
$$\simeq -9.5999 \quad (d=3). \tag{3.4}$$

Essam and Hunter [15] estimated the individual amplitudes C_4^+ and C_4^-, but did not examine the ratios. Their estimates yield $C_4^+/C_4^- = -125.6 \pm 4.0$ for the square lattice and -156 ± 10 for the triangular lattice: these are in poor accord with universality. For the bcc and fcc lattices, however, the values lead to $C_4^+/C_4^- = -9.0_5 \pm 0.5$, which we suspect are more reliable because of their larger coordination numbers. However, the sc and dia lattices suggest $-9.7_7 \pm 0.5$. The case for universality is again not good. But one might conclude $C_4^+/C_4^- = -9.4 \pm 0.8$ for $d=3$. (We again give the full uncertainties.)

Fortunately, the low-temperature series, which are frequently poorly behaved (the critical point lying *outside* the circle of convergence for sc, bcc, and fcc lattices), were greatly extended by Sykes *et al.* in 1973 [22]. Series in powers of $u = \exp(-4J/k_B T)$ can be derived for $\chi_4(T)$ to orders u^{20}, u^{28}, and u^{40} for the sc, bcc, and fcc lattices, respectively. In $d=2$ dimensions the square lattice series run to order u^{11}, the triangular series to u^{16}.

Following Liu and Fisher [7], we analyzed the amplitude series

$$\mathcal{C}_4(T) = [1-(u/u_c)]^{\gamma+2\Delta}(\partial^2\chi/\partial h^2)_0 \tag{3.5}$$

for $T \leqslant T_c$ and, similarly, above T_c with u replaced by v. As already observed, the DA technique with imposed critical point can reveal both the correction exponent θ and the desired critical amplitudes $C_4^{\pm} \propto \mathcal{C}_4(T_c \pm)$. However, for both $T \geqslant T_c$ and $T \leqslant T_c$ the correction amplitudes are sufficiently small that they cannot be detected. The generated estimates for θ are widely scattered and not significantly correlated with the amplitude estimates. The latter, however, prove to be quite sharply distributed. Accordingly, we have based our estimates on many high-order DAs, taking the two central quartiles of the distribution [12] as a basis for our estimate.

Our conclusions are collected in Table I. For reasons discussed below in Sec. V, the central estimate for C_4^+ on the sc lattice has been displaced. The individual amplitude ratios for the bcc and fcc lattices are $C_4^+/C_4^- = -8.9 \pm 0.8$ and -8.9 ± 0.2, respectively. As in the case of the Essam-Hunter estimates, the magnitude of the sc amplitude ratio is larger: $C_4^+/C_4^- = -9.1 \pm 0.8$. However, the discrepancy is now smaller, 0.2 in place of 0.7, and the shift in the sc lattice with the longer series is larger. This supports the view that the bcc and fcc results are probably more reliable. Nevertheless, all three estimates are consistent within the apparent uncertainties.

The stability of the amplitude ratios to changes in the exponent assignments was checked by computations using $\nu = 0.6335$ and $\gamma = 1.2390$ [10] in place of the Table I values. Likewise critical point shifts within the quoted uncertainties were examined. No significant changes in the final estimates result. Our overall estimate of the universal ratio for $d=3$ is given in Table III.

As regards the series for the planar lattices, only direct Padé approximants to the amplitude functions were calculated since singular corrections to scaling are not expected to play a significant role [23]. The behavior is found to be very good; our estimates are listed in Table I. Universality is well supported with the value of C_4^+/C_4^- listed in Table III.

Finally, it is interesting to compare the values in Table III with the mean-field value (3.1) and the special linear model values (3.4). The trend with d is correctly reproduced, but the linear model results are too large by factors 1.07 and 9.5 in $d=3$ and 2 dimensions, respectively, the vast discrepancy for $d=2$ being, again, not so surprising.

IV. THIRD-ORDER AMPLITUDE RATIO

The amplitudes C_3^- on various lattices were also examined by Essam and Hunter [15]. However, it seems worthwhile to update them so that the universality of R_3 may be checked. The low-temperature expansions for $(\partial\chi/\partial h)_0$ can be derived easily from Ref. [22]. Following the previous sections, we have analyzed the amplitude series

$$\mathcal{C}_3(T) = [1-(u/u_c)]^{\gamma+\Delta}(\partial\chi/\partial h)_0. \tag{4.1}$$

First we have used the DA method, with the critical exponent and temperature imposed, to study the correction exponent θ and estimate the amplitude C_3^-. However, the correction terms are again too small to be detected. Nevertheless, the C_3^- distribution is fairly sharp. Since the correction terms appear negligible, direct Padé approximants are probably reliable. We have examined the last three orders of the near-diagonal Padé approximants. The convergence proves rather good for all lattices, the results being more precise than, but in complete agreement with, the DA results. The resulting estimates are given in Table I.

To estimate the universal ratio R_3, the amplitudes B and C^- are also required. For the square lattice the former is known exactly, the latter very precisely. On the other hand, only series estimates seem available for the triangular lattice amplitude C^-, although the amplitude B is again known exactly. To find C^- for the triangular lattice, we have used the long high-temperature series, now available to update estimates for the amplitude C^+. The near-diagonal Padé approximants yield $C^+ = 0.9242_2 \pm 0.0002$, which is also confirmed by the DA technique. Then, C^- can be derived via the precisely known universal ratio for $d=2$ (see Table III). From Table I, we find

$$R_3 = \begin{cases} 33.05 \pm 0.94 & (\text{sq}) \\ 33.06 \pm 0.26 & (\text{tri}). \end{cases} \tag{4.2}$$

As one can see, universality is surprisingly well confirmed by the central values so one can be confident of the estimate for $d=2$ given in Table III.

For $d=3$ amplitudes B and C^+ we may, as discussed in Ref. [11], interpolate the results of Liu and Fisher [7] since they considered several sets of critical-exponent assignments. The entries for the sc and bcc lattices are quoted from Ref. [11], whereas the fcc amplitudes are derived by interpolation in the same manner. The amplitude C^-, then, can be derived via the universal ratio $C^+/C^- = 4.95 \pm 0.15$ [7]. In $d=3$, the third-order amplitude ratios on various lattices are

$$R_3=\begin{cases}6.43\pm 0.33 & (\mathrm{sc})\\ 6.46\pm 0.58 & (\mathrm{bcc})\\ 6.51\pm 0.49 & (\mathrm{fcc}).\end{cases} \tag{4.3}$$

Again, universality is well confirmed. For the overall estimate given in Table III, we have assigned greater weight to the bcc and fcc results.

V. RENORMALIZED COUPLING CONSTANTS

To calculate the renormalized coupling constants g_+^* and g_-^* defined in (1.8) and (1.10), the susceptibility and correlation-length amplitudes $C^\pm$ and $f_1^\pm$ are needed in addition to C_3^- and $C_4^\pm$. In the mean-field limit, hyperscaling is violated unless one takes $d=4$. Even then universality is not expected. However, following Tarko and Fisher [24,25] we can identify $\xi_1^2(T)$, on a simple hypercubic lattice, as $\chi a^2/2d$ with the result $f_1^+=1/\sqrt{2d}$. Using this and the appropriate amplitudes obtained from the Ising mean-field equation of state [26] shows that g_+^* and g_-^* have the same value and are, as expected, positive (see Table III). However, the ratio G_1^- takes a very large negative value.

One may calculate the $\epsilon=4-d$ expansion for the nonuniversal amplitudes from the equation of state and the spin-spin correlation function given in Refs. [27]: this yields

$$g_+^*=g_-^*=S^{-1}(d)[\tfrac{2}{3}\epsilon+O(\epsilon^2)], \tag{5.1}$$

$$G_1^-=S^{-1}(d)[-\tfrac{16}{3}\epsilon+O(\epsilon^2)], \tag{5.2}$$

where

$$S(d)\equiv 2\pi^{d/2}/(2\pi)^d\Gamma(d/2). \tag{5.3}$$

The ϵ expansion for the product $S(d)g_+^*$ is known to order ϵ^3 [27] and agrees with (5.1). On expanding $S(d)$ in powers of ϵ, our g_+^* calculation is also consistent with the result stated in [19]. It is remarkable that g_-^* has the same expansion as g_+^* to order ϵ. However, higher-order terms will disagree since the series analysis for $d=2$ reveals a large difference. It should also be noticed that the ϵ expansions predict that $g_\pm^*$ and G_1^- vanish at $d=4$, whereas the mean-field theory gives nonzero finite values. This is because, as ϵ goes to zero, the Wilson-Fisher Ising fixed point in the space of Hamiltonians approaches the Gaussian fixed point. Even though the latter reproduces the mean-field exponents, it does not fully represent the mean-field fixed point. For example, the spontaneous magnetization amplitude B is $\sqrt{3}\mu_0$ from the Ising mean-field equation of state [26], but the ϵ expansion for B diverges in leading order as $\epsilon^{-1/2}$ [19].

Upon using (1.9) and the amplitude estimates collected in Table I, we find the renormalized coupling constant values

$$g_+^*=\begin{cases}14.699\pm 0.014 & (\mathrm{sq})\\ 14.702\pm 0.017 & (\mathrm{tri})\end{cases} \tag{5.4}$$

for $d=2$ and

$$g_+^*=\begin{cases}24.55^{+0.95}_{-0.25} & (\mathrm{sc})\\ 24.39\pm 0.09 & (\mathrm{bcc})\\ 24.50\pm 0.13 & (\mathrm{fcc})\end{cases} \tag{5.5}$$

for $d=3$. The universality across lattices of the same dimensionality is well confirmed. (Note, however, that for the sc lattice we use the displaced ‘‘central’’ estimates for C_4^+ and f_1^+ in Table I, since these bring the central estimate for g_+^* closer to the bcc and fcc ranges, which are both significantly smaller.) Table III presents our overall estimates. The field-theoretic estimates [3,4] agree well with our series estimates in both dimensions as do Baker's early series estimates [18]. However, our estimates are more precise.

The amplitude ratio G_1^- can be calculated by combining the universal amplitude ratios C^+/C^-, C_4^+/C_4^-, and f_1^+/f_1^- with g_+^* (see Table III). In addition, the renormalized coupling constant g_-^* is readily calculated via (1.11). The estimates for C^- and f_1^- are derived from the corresponding amplitudes above T_c and the universal ratios C^+/C^- and f_1^+/f_1^- given in Table III. In Table I, only the *true* correlation-length amplitude is listed in $d=2$. However, using the estimated ratio f^+/f_1^+ [25,26], one can obtain f_1^+ and hence f_1^-. We find

$$g_-^*=\begin{cases}25.4\pm 113.9 & (\mathrm{sq})\\ 21.2\pm 24.6 & (\mathrm{tri})\end{cases} \tag{5.6}$$

for $d=2$ and

$$g_-^*=\begin{cases}87.5^{+86.5}_{-68.5} & (\mathrm{sc})\\ 83.5\pm 81.8 & (\mathrm{bcc})\\ 86.5\pm 59.2 & (\mathrm{fcc})\end{cases} \tag{5.7}$$

for $d=3$. The very large uncertainties arise from the near cancellation in (1.11). However, despite this, the central estimates lie close to each other, so we suspect the uncertainties assigned are too conservative. The average values are given in Table III as our best estimates, but without quoted uncertainties.

In order to gauge the uncertainties better, we have studied the series $[(\partial^2\chi/\partial h^2)-3(\partial\chi/\partial h)^2/\chi]/\chi$ as well as the low-temperature series expansion for $g_-^*(T)$ itself. It transpires, however, that neither of these gives better results when using DA techniques. Furthermore, the correction-to-scaling terms indicated by DA methods are non-negligible so that direct Padé approximants also do not yield satisfactory estimates.

Most of the universal amplitude ratios in Table III exhibit a monotonic trend with dimensionality d: the exceptions are the hyperscaling ratios $g_\pm^*$ and G_1^-. Even though the individual amplitudes defining $g_\pm^*$ and G_1^- exhibit parallel trends, the correlation-length amplitudes $f_1^\pm$ enter to the powers of d and this proves sufficient to destroy the overall monotonicity.

VI. SUMMARY

Using modern estimates (or exact values) of critical points and exponents for $d=2$ and 3 Ising systems, existing series expansions for various lattices have been analyzed to estimate the renormalized coupling constants $g_\pm^*$ and related universal amplitude ratios (see Table III). Our estimates for g_+^* agree completely with previous ϕ^4 field theory results but are more precise. Owing to the cancellation of large values via subtraction, we were unable to gauge the uncertainties in our g_{-}^{*} estimates reliably. However, the central estimates for the sc, bcc, and fcc lattices lie close together so that the average values should serve as reasonable estimates. Other amplitude ratios on various lattices, such as C_4^+/C_4^-, R_0, and R_3, confirm universality very well. The agreement with the linear model predictions for $d=3$ is quite good.

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